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Λ-TRANSFER AND HARSANYI NTU VALUES : INDIVIDUAL RATIONALITY, STABILITY AND DEGENERATE SOLUTIONS[1]

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We investigate the properties of individual rationality and stability for Harsanyi and λ-transfer value for non transferable utility games, paying particular attention to degenerate solutions. We stress some situations in which the two values behave differently. We also give a general existence theorem for Harsanyi value, which does not require the assumption of games being compactly generated.

1. Introduction

MODERN GAME THEORY is particularly interested in giving well-founded definitions of solutions of games : we see on the one hand the refinement program in non cooperative theory and on the other hand the development of a number of solution concepts for cooperative games. Within the cooperative theory, some recent papers gave axiomatic characterizations of solutions (see, for instance, [2], [7], [13], [15], [19], [26], [27], [28]). This approach is interesting because it sheds light on the structural properties of solution concepts and emphasizes differences among them(6). Among cooperative solutions, λ-transfer (or Shapley NTU) and Harsanyi values for non transferable utility games were recently object of a deep study (see [2], [7], [9], [11], [15] and [1], [3], [22], [24]).

In this paper we join this literature, by providing an analysis of degenerate solutions and of the properties of individual rationality and stability. We conclude the analysis with an existence theorem. In what follows our results are described in more detail.

a) Degenerate solutions

Both values are attached with some comparison weights for the players that can be seen as a rescaling of their utilities. However this interpretation is justified only if they are all strictly positive. But ([25] p. 260) "zero weights can occur, and must be allowed if the existence theorem is to hold in general". Degenerate solutions,

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(1) This is a substantially revised version of a working paper by the same authors.

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(6) We recall that it was already used in the characterization of bargaining solutions and of the Shapley value (for TU games) (see for instance [21] and [20]).

namely those to which weights are attached that can vanish for some player, are an aspect of a lively debate between Aumann and Roth, ([1], [3], [24] and references therein quoted), proving that their role is not uncontroversial : "their intuitive significance is murky" ([2] p. 269), though ([3], p. 985) (zero weights)" are indeed associated with degeneracies, but they are in the game, not in the value". Here we try to give a contribution to the understanding of their role, providing a simple geometric characterization of the case when they occur. This in turn gives a sufficient condition, that can be naturally interpreted as a superadditivity condition, to rule out the possibility of zero weights; furthermore our condition, unlike ones previously suggested, is not related to the geometric shape of the set of outcomes for the grand coalition. Degenerate solutions are next studied also from the point of view of individual rationality.

b) Individual rationality

A selected outcome of a game satisfies this property if every player receives at least what he is able to get acting by himself. It really seems a very minimal and natural property to be fulfilled : one reason why Harsanyi changed his previous definition of value [4], was that it did not satisfy individual rationality; moreover it is almost always present in bargaining solutions [21] and it is frequently used as an axiom to characterize cooperative solutions. Under some (weak) superadditivity conditions, it is not difficult to see that both values, when attached with strictly positive weights, are individually rational. The situation changes dramatically for degenerate solutions; moreover the two values behave much differently. We show that a degenerate Shapley value may not be individually rational and, at the same time, the only one corresponding to that weight : we observe that this feature seems to be really hard to justify. As already mentioned, the situation changes for the other value : also a Harsanyi degenerate solution may not be individually rational, but every intermediate agreement among coalitions leading to that solution does satisfy the property; more importantly, we show that to the same weights and to the same configuration of agreements among coalitions one can always associate another individually rational value. Therefore we suggest the idea of rejecting every non individually rational value : it can be substituted by a rational one, attached to the same bargaining mechanism and rescaling factor. As a consequence, even degenerate Harsanyi values can be perfectly acceptable, at least from this point of view.

c) Stability

Stability of equilibria with respect to perturbations is primarily a mathematical property, but it can sometimes have importance also from a purely game theoretical point of view; we find it as a property defining the axiomatic framework for certain cooperative solutions : [13] [27] [28 and related references]; even more, it plays a central role in many refinement concepts (robustness with respect to "trembling") : consider as an example the various stable sets in the very stimulating paper by Kohlberg and Mertens [17]. Stability for games was already considered in [12] for bargaining solutions, and in [14] and [18] for many cooperative concepts. Here we do the work for the two values. The results we offer, roughly speaking, are closedness results, in the following sense : given a sequence of games converging to a limit game, every cluster point of values for the sequence is a value for the limit game. Moreover we show that, even for very special subclasses of games, lower semicontinuity fails : this means that a solution of a game cannot be approximated by values of games which are close to it. It can be remarked that, in general, these are exactly the results one could expect from solutions of optimization problems. We agree, and this is the reason why we want to stress here that there are important differences between the two values. The Shapley value is *not* graph closed with respect to the most natural topology on games one can think of, while Harsanyi value does. We are able to derive a graph closedness result for the λ-transfer value only with a much more restrictive convergence on a restricted class of games. This has also a relevant consequence on general existence theorems, as we shall see. We now say a few words on the topology we consider on the set of NTU games, viewed as (closed) subsets of an Euclidean space. We shall use Kuratowski (otherwise called closed) convergence extensively, even though Hausdorff convergence seems to be more popular in this setting (see [9], [13], [27], [28]); nevertheless our choice is motivated by the following reasons : we shall deal with sets that are always unbounded and not necessarily compactly generated; hence Hausdorff convergence is, in this context, too restrictive. Moreover, from a topological point of view, closed convergence is equally appealing, as it can be shown that it is induced by a metric for which the set of closed subsets of a locally compact Hausdorff space is compact [10], [16]. To conclude about stability we therefore show a different behaviour between the two values, and we emphasize the fact that the results have a natural interpretation from a purely game theoretical point of view : namely Harsanyi value depends essentially only on the region of the individually rational payoffs : this entails enough compactness to get the result, while Shapley value can dramatically depend on payoffs that are really disagreable for the players : actually no lower bound can be a priori imposed on the utility levels taken into account.

d) An existence result

All the previous rersults are put together in order to get a general existence theorem for the Harsanyi value, along the following lines : at first, it is a known result that convex compactly generated games have a value [20]. Next we use closed convergence in order to approximate a given general game by a sequence of compactly generated ones. Individual rationality, even of degenerate solutions, provide enough compactness to ensure the existence of cluster points for every sequence of values. Finally, stability implies that these cluster points are values for the game we start with. If we need to avoid zero weights, we only have to impose the superadditivity condition which, in the analysis of degenerate solutions, was shown to provide only strictly positive weights. This simple general and classical procedure cannot be used to get the same result for the λ-transfer value. First of all, a general game cannot usually be approximated in Hausdorff sense with compactly generated ones; secondly, individual rationality does not provide compactness because of the degenerate case; finally, stability is in any case established on a rather restricted class. This clearly explains why in [15] a careful work had to be done in order to select a *particular* approximating (in Kuratowski sense) sequence, to show stability *along this sequence,* to ensure positive weights at every step to get the necessary compactness. Compare it with our few lines proof ! Essentially the theorem summarizes the differences between the values we have investigated here.

The rest of the paper is organized as follows : section two provides the necessary preliminaries and the definitions of the values, section three deals with degenerate solutions, considered also in section four where individual rationality is investigated. Sections five and six analyze stability, section seven presents the existence theorem, while in section eight we make some final comments.

2. Preliminaries

Denote the real numbers by $\mathbb{R}$ and the positive integers by $\mathbb{N}$. In any game we shall consider the set of players fixed and denoted by N; we suppose, without loss of generality, $N = \{1, \ldots, N\}$. Nonempty subsets S of N are called coalitions; 2^N indicates the set of all coalitions, while the number of players in S is denoted by $|S|$. We identify $\mathbb{R}^N$, the space of functions from N to $\mathbb{R}$, with the Euclidean space $\mathbb{R}^N$ and, for $S \in 2^N$, $\mathbb{R}^S$ will be identified with the subset of $\mathbb{R}^N : = \{x : x^i = 0 \text{ if } i \notin S\}$. By $\mathbb{R}_+^S$ ($\mathbb{R}_-^S$) we denote the set of vectors x such that $x^i \geq 0$ ($x^i \leq 0$) for all $i \in S$. For a vector $x \in R^n$, x_S will be the element of R^S with the i^{th} coordinate x^i, for each $i \in S$, but we shall write x^i rather than $x_{\{i\}}$; the L^∞-norm of x is $|x|_\infty : = \max \{|x^i| : i \in N\}$. For $x \in \mathbb{R}^S$ and $y \in R^T$ in an obvious way $x + y$ can be seen as a vector of $\mathbb{R}^{S \cup T}$. The standard inner product between two vectors x and y is denoted by $x \,.\, y$, while xy is the vector such that $(xy)^i = x^i y^i$ for every i. We shall write $x \leq y$ ($x < y$) to intend $x^i \leq y^i$ ($x^i < y^i$) for every i.

Now we recall the notions of set-convergence we shall use in the sequel. (see for instance [10] and [16]) :

DEFINITION 2.1. Let V_m, $m \in \mathbb{N}$, be a sequence of (closed) sets in $\mathbb{R}^n$. We say that $V_m \overset{K}{\rightarrow} V_0$ in Kuratowski sense (or for the closed convergence) if :

(2.1) $V_0 \subset \liminf V_m := \{x : \exists\, x_m \rightarrow x,\ x_m \in V_m \text{ for all large } m\}$

(2.2) $V_0 \supset \limsup V_m := \{x : \exists\, x_k \rightarrow x,\ x_k \in V_{m_k},\ m_k \text{ a selection from } \mathbb{N}\}$.

Although not always, property (2.1) is often called lower semicontinuity and (2.2) graph closedness. We shall use always this terminology here.

DEFINITION 2.2. We say that the sequence of (closed) nonempty sets V_m converges to V_0 in Hausdorff sense, written $V_m \overset{H}{\rightarrow} V_0$ if :

(2.3) $\forall \varepsilon > 0\ \exists k \in \mathbb{N} : \forall m > k\ V_m \subset B_\varepsilon(V_0)$

(2.4) $\forall \varepsilon > 0\ \exists k \in \mathbb{N} : \forall m > k\ V_o \subset B_\varepsilon(V_m)$,

where $B_\varepsilon(V) = \{x : \exists y \in V : |x-y|_\infty < \varepsilon\}$.

It is straightforward to check that given a set of closed sets in a metric space, Hausdorff convergence on it is more restrictive than closed convergence.

A cooperative (NTU) game is a map which assigns to every coalition S a non-empty subset of $\mathbb{R}^S$, with some geometrical structure. Here we shall use the following set of assumptions on the set of games, that we indicate by Γ :

(G.1) Closedness : V(S) is a closed set $\forall S$.

(G.2) Comprehensiveness : $V(S) \supset V(S) + \mathbb{R}^S_-$ $\forall S$.

(G.3) Boundedness : there are $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $a > 0$ and $V(N) \subset \{x \in \mathbb{R}^n : a \,.\, x \leq c\}$.

(G.4) Weak superadditivity : $V(S \cup \{i\}) \supset V(S) + V\{i\})$, $\forall S$, $\forall i \notin S$.

Observe that (G.1) (G.2) and (G.3) force $V(\{i\}) = (-\infty, v(i)]$, as a subset of $\mathbb{R}^i$, for some real number v(i).

A sidepayment game is a function $v: 2^N \rightarrow \mathbb{R}$. If $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ $\forall S$, $\forall i \notin S$, then the sidepayment game can be imbedded in the set Γ of NTU games by defining : $V(S) := \{x \in \mathbb{R}^S : \sum_{i \in S} x^i \leq v(S)\}$.

We shall indicate by $\hat{\Gamma}$ the subset of Γ of compactly generated games, i.e. $V \in \hat{\Gamma}$ if $V \in \Gamma$ and for every S there exists a compact set K_S such that $V(S) = K_S + \mathbb{R}^S_-$.

The set Γ can naturally be considered as a collection of closed subsets of an Euclidean space $E(E = \prod_{S \in 2^N} \mathbb{R}^S$ for instance), namely as a subset of the hyperspace of the closed sets of E. In this space, closed convergence is compatible with a metric, that makes compact the hyperspace ([10], [16]). In the sequel, we shall speak of closed or Hausdorff convergence of games to intend that, for every coalition S, the sequence of outcomes converges in the corresponding sense. Observe that a sidepayment game cannot be approximated in Hausdorff sense with a sequence of compactly generated games, while this can easily be done with closed convergence.
It is now time to introduce the values of games in Γ we want to investigate. To this end, let us recall that the Shapley values $\phi(v)$ of the sidepayment game v is the vector in $\mathbb{R}^n$ defined by

$$(2.5) \qquad \phi^i(v) = \sum_{S \in 2^N, i \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} (v(S) - v(S - \{i\})).$$

Define, for $\lambda \in R^n_+ - \{0\}$ and $V \in \Gamma$, the corresponding sidepayment game v_λ as :

$$(2.6) \qquad v_\lambda (S) := \sup \{\lambda \, . \, x : x \in V(S)\}, \text{ for } S \in 2^N.$$

DEFINITION 2.3. A vector $y \in V(N)$ is called a Shapley (or λ-transfer) value for $V \in \Gamma$ if there is a vector λ such that :

(S.1) $\quad v_\lambda (S) < +\infty \ \forall S$

(S.2) $\quad \lambda y = \phi(v_\lambda)$.

The set $\Phi(V) := \{y \in V(N): y$ is a Shapley value for $V\}$ is called the Shapley value (set) of V and $\Phi: \Gamma \to \mathbb{R}^n$ is the Shapley (value) multifunction.
We now turn to the Harsanyi value. Call an element $\hat{x} = (x_S)_{S \in 2^N}$ a payoff configuration.

DEFINITION 2.4. An element $h \in V(N)$ is said to be a Harsanyi value for $V \in \Gamma$ if there is a vector λ, a payoff configuration $\hat{h}$ with $h_N = h$ and real numbers ξ_S, $\forall S$, such that :

(H.1) $\quad h_S \in bd(V(S)) \; \forall S$

(H.2) $\quad \lambda . h \geq \lambda . x \quad \forall x \in V(N)$

(H.3) $\quad \lambda^i h_S^i = \sum_{T \subset S, i \in T} \xi_T \quad \forall i \in S,$

where bd (V) indicates the boundary of the set V.
$\mathbb{H}: \Gamma \rightarrow \mathbb{R}^n$ will be the related Harsanyi multifunction; observe that both $\Phi(V)$ and $\mathbb{H}(V)$ can be empty valued for some V.

DEFINITION 2.5. A value is called degenerate if it corresponds to a weight vector λ with some $\lambda^i = 0$.

For a vector $\lambda > 0$ there is at most one Shapley value; this can be false if λ has some zero coordinates. For a Harsanyi value, with $\lambda > 0$, the corresponding configuration and coefficients are uniquely determined. This follows easily from (H.4) and (H.5) below : let h be a value $\hat{h}$ and ξ_S associated to it. Put $t_{\{i\}} = 0$ for $i \in N$ and $t_S = \sum_{T \subset S} (-1)^{(|S| - |T| - 1)} h_T$. Then :

(H.4) $\quad \lambda^i(h_S^i - t_S^i) = \lambda^j(h_S^j - t_S^j) = \xi_S \quad \forall i,j \in S, \forall S.$

(H.5) $\quad \lambda . h_S = \max \{\lambda . x : x \in V(S), \lambda^i(x^i - t_S^i) = \lambda^j(x^j - t_S^j)\}, \forall i,j \in S, \forall S.$

Every book of game theory can illustrate the assumptions on games we deal with here; see for instance [20]; moreover we refer to [2], [7], [9], [11], [15] for extensive discussions of the meaning and of the properties of these two values.
To finish we quickly describe the properties we shall investigate. Let $v = (v(\{1\}, \dots v(\{n\}))$. Then a vector $x \in V(S)$ will be called individually rational if $x \geq v_S$. Hence a value h is individually rational if $h \geq v$, while a payoff

configuration $\hat{h}$ is called individually rational if $h_S \geq v_S \ \forall S$. As far as stability is concerned, we shall study graph closedness with respect to closed (or Hausdorff, if closed fails) convergence, and we shall show by examples that lower semicontinuity fails even in Hausdorff sense, in the non degenerate case and inside the domain of existence of values.

3. Degenerate Solutions

Throughout this section we consider a fixed weight vector λ, which will be the vector corresponding to the solutions we shall deal with. Having in mind the investigation of degenerate solutions, some component of λ is assumed to be zero. Let us call $I := \{i \in N: \lambda^i > 0\}$ and $J = N - I$. Hence in this section J is nonempty (while I can never be empty by definition).

Lemma 3.1. Let y be a Shapley solution for V. Then for the corresponding game v_λ we have : $v_\lambda(S) = v_\lambda(I)$ for each $S \supset I$.

Proof. It is sufficient too show that

$$(3.1) \qquad v_\lambda(S) = v_\lambda(S - \{j\}) \text{ for all } S \supset I \cup \{j\} \text{ and } j \in J.$$

Take S and j as before. Because y is a Shapley solution we have :

$$(3.2) \qquad 0 = \lambda^j y^j = \sum_{S \in 2^N, j \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} (v_\lambda(S) - v_\lambda(S - \{j\})).$$

Furthermore we have by (G.4), for $j \in J$,

$$(3.3) \qquad v_\lambda(S) \geq v_\lambda(S - \{j\}) + v_\lambda(\{j\}) = v_\lambda(S - \{j\}).$$

Combining (3.2) and (3.3) we obtain (3.1).

Theorem 3.2 Let $T \supset S \supset I$. If there is $x \in V(S)$ such that $\lambda . x = v_\lambda(S)$, then for all $u \in \mathbb{R}^{T-S}$ with $x + u \in V(T)$ we have $x + u \in bd(V(T))$.

Proof. By lemma 3.1, $v_\lambda(S) = v_\lambda(T)$. If there are some x and u as described above with $x + u \in int(V(T))$, then there is a $z \in V(T)$ with $z > x + u$ and then

$$v_\lambda(T) \geq \lambda . z > \lambda . (x + u) = \lambda . x = v_\lambda(S) = v_\lambda(T),$$

a contradiction. So $x + u \in bd(V(T))$.

From the previous theorem we get the following superadditivity condition in order to eliminate zero weights.

THEOREM 3.3. Let $V \in \Gamma$ be such that for each $i \in N$ the set $V(N - \{i\})$ is compactly generated and

$$(3.4) \qquad V(N - \{i\}) + V(\{i\}) \subset \text{int}\,(V(N)).$$

Then V does not have degenerate solutions.
Proof. Suppose, on the contrary, that there is a degenerate solution corresponding to some λ with $\lambda^j = 0$. Since there exists $x \in V(N - \{j\})$ such that $\lambda_{N-\{j\}} \cdot x = v_\lambda(N - \{j\})$, by theorem 3.2 (and by (G.4)), $x + v(j)^{(7)} \in bd(V(N))$. This contradicts 3.4.

Example 3.4. Let $N = \{1,2,3\}$, $v(i) = 0$ for all i, $V(\{1,2\}) = \{(x,y,0): x,y \leq 0\}$, $V(\{1,3\}) = \{(x,0,z): x,z \leq 0\}$, $V(\{2,3\}) = \{(0,y,z): z \leq 1$ if $y \leq 1$ and $z \leq 3 - y - \frac{1}{y}$ if $y \geq 1\}$ $V(N) = \{(x,y,z) : x \leq 1, y + z \leq 3\}$.

An easy computation shows that $y = (0, 1.5, 1.5)$ is a degenerate Shapley solution associated with $\lambda = (0,1,1)$. But (3.4) holds. The pathology here is due to the fact that $v_\lambda(\{2,3\}) = \lambda \cdot x$ for no $x \in V(\{2,3\})$.

Now we consider degenerate Harsanyi solutions.

Lemma 3.5. Let h be a payoff configuration associated to the Harsanyi solution h. Let $S \in 2^N$ be such that $S \supset I$. Then, for all $T \in 2^N$

$$(3.5) \qquad h_T^i = h_{S \cap T}^i \text{ for all } i \in I \cap T$$

holds.
Proof. First of all, it is easy to show that, if $j \in J$, then $\xi_T = 0$ for all the coalitions T such that $j \in T$. This can be made for instance by induction on the cardinality of the coalitions and remembering (H.3). Now we prove (3.5). Note first that (3.5) holds if $T \supset S$. So, let us suppose that $\phi \neq T - S$. Then by (H.3) and the remark made before

(7) Here, by abuse of notation, $v(j)$ indicates the vector y of $\mathbb{R}^n$ such that $y_i = 0\ \forall\, i \neq j$, $y_j = v(j)$.

(3.6) $\lambda^i h^i_T = \sum_{i \in U \subset T} \xi_U = \sum_{i \in U \subset T \cap S} \xi_U = \lambda^i h^i_{T \cap S}$. Since $\lambda^i \neq 0$ if $i \in T \cap I$, we get (3.5) from (3.6).

THEOREM 3.6. Let $\hat{h}$ be a payoff configuration associated to h and let $I \subset S \subset T$. Then, for each $x \in \mathbb{R}^{T-S}$ with $h_S + x \in V(T)$, we have $h_S + x \in bd(V(T))$.

Proof. Suppose, on the contrary, that there is $z \in V(T)$: $z > h_S + x$. In view of lemma 3.5 we have that $h^i_T = h^i_S$ for all $i \in I$. Let t be the vector in $\mathbb{R}^{N-T}$ whose i-th component is v(i), $i \notin T$. Then $z + t \in V(N)$ and $\max_{a \in V(N)} \lambda \cdot a = \lambda \cdot h_S < \lambda \cdot (z+t) \leq \max_{a \in V(N)} \lambda \cdot a$, a contradiction.

THEOREM 3.7. Let $V \in \Gamma$ be such that for all $i \in N$ the inclusion (3.4) holds. Then V does not have degenerate Harsanyi solutions.

Proof. See theorem 3.3.

To conclude the section we want to stress the meaning of theorem 3.6 on degenerate Harsanyi solutions : no coalition containing the players associated to positive weights can improve its payoff (for each player) by cooperating with the players associated with zero weights(8)

4. Individual rationality

For a non degenerate Shapley solution it is easy to verify that it is individually rational : it follows directly from the fact that v_λ is superadditive in view of (G.4) and formula (2.5). But in general degenerate Shapley solutions do not have this property, as the following example shows :

EXAMPLE 4.1. Let $N = \{1,2,3\}$, $v(i) = 0$ for all $i \in N$, $V(\{1,2\}) = \{(x,y,0)$: $x,y \leq 0\}$, $V(\{1,3\}) = \{(x,0,z)$: $x,z \leq 0\}$, $V(\{2,3\}) = \{(0,y,z)$: $y \leq 1$, $z \leq 0\}$, $V(N) = \text{conv}\,(\{(0,1,0), (-1,0.5, 0.5)\}) + \mathbb{R}^N_-$.

The *unique* solution associated to $\lambda = (0,1,1)$ is $y = (-1,0.5,0.5)$ which is not individually rational. Observe that $V \in \hat{\Gamma}$ and that V(S) is convex for each $S \in 2^N$.

(8) These results suggest the idea that weights represent also the bargaining power of a player. Namely a "zero weight" player does not improve utility levels of all players (at the same time) in any coalition, by cooperating with them. For Shapley value a similar interpretation holds (cf. [3] p. 988), but example 3.4 shows an important difference. In any case zero weight does not mean that it is really a dummy player, as it is argued in [24] p. 983.

To study Harsanyi solutions, we fix λ and we consider the sets I and J as in the previous section; however now J can be empty.

THEOREM 4.2. Let $\hat{h}$ be a payoff configuration associated to some Harsanyi solution h. Then

$$(4.1)\qquad h^i_S \geq v(i) \text{ for every } i \in I \text{ and for every } S \subset N.$$

Proof. From lemma 3.5 it is enough to show the claimed property for each $S \subset I$. From (H.1) it follows that $h_{\{i\}} = v(i)$. Now suppose (4.1) holds for each coalition $S \subset I$ with $|S| \leq s$: have to show that (4.1) is true for the coalitions T with $|T| = s + 1$. Suppose w.l.o.g. $T = \{1, \ldots, s + 1\}$. If (4.1) does not hold for T we can suppose, for instance,

$$(4.2)\qquad h^1_T < v(1).$$

We claim that in this case

$$(4.3)\qquad h^i_T < h^i_{\{2,\ldots,s+1\}} \text{ for each } i \in \{2, \ldots, s+1\}.$$

Suppose (4.3) is not true. Then w.l.o.g.

$$(4.4)\qquad h^2_T > h^2_{\{2,\ldots,s+1\}}\,.$$

By (H.3), $\lambda^2 (h^2_T - h^2_{\{2,\ldots,s+1\}}) = \sum_{2\in S\subset T} \xi_S = \sum_{2\in S\subset T, 1\in S} \xi_S$.

Then, by (4.4)

$$(4.5)\qquad \sum_{\{1,2\}\subset S\subset T} \xi_S \geq 0.$$

By (4.2) we have $0 > \lambda^1 (h^1_T - v(1)) = \sum_{1\in S\subset T} \xi_S - \xi_1$, or

$$(4.6) \qquad \sum_{1 \in S \subset T, S \neq \{1\}} \xi_S < 0.$$

Furthermore,

$$\lambda^1 h^1_{\{1,3,\dots,s+1\}} = \sum_{1 \in S \subset \{1,3,\dots,s+1\}} \xi_S = \sum_{1 \in S \subset T} \xi_S - \sum_{\{1,2\} \subset S \subset T} \xi_S \leq \sum_{1 \in S \subset T} \xi_S = \xi_{\{1\}} + \sum_{1 \in S \subset T, S \neq \{1\}} \xi_S < \xi_{\{1\}} = \lambda^1 v(1),$$

where the first inequality follows from (4.6). Hence,

$$(4.7) \qquad h^1_{\{1,3,\dots,s+1\}} < v(1).$$

But (4.7) contradicts the inductive hypothesis. So (4.3) holds if (4.2) is true. Now let $x = (v(1), h^2_{\{2,\dots,s+1\}}, \dots, h^{s+1}_{\{2,\dots,s+1\}})$.

By (G.4), $x \in V(T)$. But (4.2) and (4.3) together imply that x strictly dominates h_T, which contradicts (H.1). Hence (4.1) also holds for T. This completes the proof.

THEOREM 4.3. Let h be a Harsanyi solution with $\hat{h}$ an associated payoff configuration. Then there is a payoff configuration $\hat{y}$ (9) associated to h, such that $y^i_S \geq v(i)$ for each i and for each $S \neq N$; moreover there is a payoff configuration $\hat{z}$ such that $z^i_S \geq v(i)$ for each i and S.

Proof. Let $z^i_S = h^i_S$ for each $i \in I \cap S$, $z^i_S = \max(v(i), h^i_S)$ for each $i \in S\text{-}I$.

Let $y_S = z_S$ for each $S \neq N$, $y_N = h$.

Observe that $z_S \in V(S)$ ($y_S \in V(S)$) because $z_S(y_S)$ is a vector on $\mathbb{R}^S$ of the kind $h_T + v_{S-T}$ for some $t \subset S$. Theorem 4.2 shows that $z^i_S \geq v(i)$ if $i \in I$. Then individual rationality of z_S for all S and of y_S for all $S \neq N$ is a trivial matter. (H.2) and (H.3) for $\hat{z}$ easily follow from the fact that $\lambda^i z^i_S = \lambda^i h^i_S$ for all i and S. (H.1) is an immediate consequence of the fact that $z_S \geq h_S$ and $h_S \in bd(V(S))$.

(9) Associated to the same weight vector λ.

5. Stability of the Shapley value

We start with the following :

EXAMPLE 5.1. Let $N = \{1,2,3\}$ and let V_m be the sequence of games in Γ defined by : $v_m(i) = 0$ for $i \in N$, $V_m(\{1,2\}) = \{(x,y,0) : x,y \leq 0\}$, $V_m(\{1,3\}) = \{(x,0,z) : x,z \leq 0\}$, $V_m(\{2,3\}) = \{(0,y,z) : y \leq 0, y + z \leq 1, y + \frac{m}{m+1} z \leq 0\}$; $V_m(N) = \{(x,y,z): x + y + z \leq 2\}$.

This sequence of games converges in Kuratowski's sense to the game V such that $V(S) = V_m(S)$ for $S \neq \{2,3\}$ and $V(\{2,3\}) = \{(0,y,z): y \leq 0, y + z \leq 0\}$.
A straightforward calculation shows that $\Phi(V_m) = (\frac{1}{3}, \frac{5}{6}, \frac{5}{6})$ and $\Phi(V) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, with $\lambda = (1,1,1)$.
From this example we learn that, in general, the Shapley value multifunction does not satisfy any stability property, at least with respect to Kuratowski convergence; we can also notice that $(v_\lambda)_m(\{2,3\})$ does not converge to $v_\lambda(\{2,3\})$, and this is the reason for the pathological phenomenon displayed here.

The question arises whether there are subsets of Γ on which there is a more regular behaviour of Φ.

For the question of closedness we deal with a sequence of games V_m converging to a limit game V, a sequence $y_k \in \Phi(V_{m_k})$ converging to some vector y and a sequence of weights λ_k which, w.l.o.g., we can suppose to converge to some $\lambda \in \mathbb{R}^n_+$. The question is whether y is a Shapley solution for the game V(with λ as weight vector). By the continuity of the Shapley value for sidepayment games it follows immediately that $\lambda y = \Phi(v_\lambda)$ *if* $(V_{\lambda_k})_{m_k}(S)$ converges to $v_\lambda(S)$ for all S. It is straightforward to check that (even) closed convergence implies

$$\liminf (v_{\lambda_k})_{m_k}(S) \geq v_\lambda(S) \text{ for all } S \in 2^N.$$

so, to obtain a closedness result, we have to find conditions guaranteeing that

$$\limsup (v_{\lambda_k})_{m_k}(S) \leq v(S) \text{ for all } S \in 2^N. \tag{5.1}$$

THEOREM 5.2. Let V_m be a sequence of games Hausdorff converging to the *compactly generated* game V. Then $\limsup \Phi(V_m) \subset \Phi(V)$.

Proof. We have to prove that (5.1) holds. Take $\varepsilon > 0$.

Observe that for each $\lambda \geq 0$ with $\sum_{i=1}^{n} \lambda^i = 1$, we have

$$\sup_{x \in B_\varepsilon(V(S))} \lambda . x \leq \sup_{x \in V(S)} \lambda . x + \varepsilon.$$

Now suppose that $V(S) = K_S + \mathbb{R}^S_-$ with K_S a compact set.

Since $\lim (\max_{x \in K_S} \lambda_k . x) = \max_{x \in K_S} \lambda . x$ we have, for large k,

$$\sup_{x \in V_{m_k}(S)} \lambda_k . x \leq \sup_{x \in B_\varepsilon(V(S))} \lambda_k . x \leq \sup_{x \in V(S)} \lambda_k . x + \varepsilon = \max_{x \in K_S} \lambda_k . x + \varepsilon \leq \max_{x \in K_S} \lambda . x + 2\varepsilon = \sup_{x \in V(S)} \lambda . x + 2\varepsilon,$$

from which we obtain :

$$\limsup (v_{\lambda_k})_{m_k}(S) = \limsup \sup_{x \in V_{m_k}(S)} \lambda_k . x \leq \sup_{x \in V(S)} \lambda . x + 2\varepsilon = v_\lambda(S) + 2\varepsilon .$$

This completes the proof.

Observe that Hausdorff convergence of V_m to V in $\hat{\Gamma}$ is a less strong condition than the convergence of compact generators of the games, as easy examples show. One further remark is that we stated the theorem on the assumption of V being compactly generated, but it is not difficult to see, using the same idea of looking for continuity of u_λ, that other geometric conditions, as for instance those one listed in [15] (lemmas 3.9 and 3.10) lead to the same result.

About lower semicontinuity of the Shapley multifunction the example 5.1 shows that in general the property is not satisfied; the next one shows that Φ does not have this property even on $\hat{\Gamma}$, equipped with Hausdorff convergence.

EXAMPLE 5.3. Let $N = \{1,2,3\}$ and let

$v_m(i) = 0$ for $i \in N$, $V_m(\{1,2\}) = \{(x,y,0): x,y \leq 0\}$, $V_m(\{1,3\}) = \{(x,0,z): x,z \geq 0,\ 24x + 72z \leq 7\} + \mathbb{R}^{\{1,3\}}_-$, $V_m(\{2,3\}) = \{(0,y,z): y,z \geq 0,\ 24y + 72z \leq 7\} + \mathbb{R}^{\{2,3\}}_-$, $V_m(N) = \{(x,y,z) \in \mathbb{R}^N_+ : 3x + y + z \leq 1, (2 + \frac{1}{m})\, x + (2 - \frac{1}{m})\, y + z \leq 1\} + \mathbb{R}^N_-$.

The sequence V_m converges in Hausdorff sense to the game V, where V is defined by $V(S) = V_m(S)$ for $S \neq N$, $V(N) = \{(x,y,z) \in \mathbb{R}_+^N : 3x + y + z \leq 1, 2x+2y+z \leq 1\} + \mathbb{R}_-^N$.

For each $m \in \mathbb{N}$ the vector $12^{-2}(\frac{55}{3}, 13, 76)$ is a Shapley solution associated to $\lambda = (3,1,1)$. No other solutions exist. This can be seen by noticing first that in view of corollary 3.3 there are no degenerate solutions. Moreover the only possible λ's which can correspond to a solution are positive multipliers of $(2 + \frac{1}{m}, 2 - \frac{1}{m}, 1)$ or $(2 + t + \frac{1 - t}{m}, 1)$ with $0 \leq t \leq 1$. Standard calculations show that for those λ's (except (3,1,1)) no solution exists. Now the limit game has also the solution 12^{-2} (17,17,76) associated to $\lambda = (2,1,1)$ as one can verify.

6. Stability of the Harsanyi value

The Harsanyi value multifunction has a closed graph, as the following theorem shows.

THEOREM 6.1. Let $V_m \in \Gamma$ be a sequence of games converging in Kuratowski sense to a game $V \in \Gamma$. Then $\limsup \mathbb{H}(V_m) \subset \mathbb{H}(V)$.

Proof. Let $h_k \in \mathbb{H}(V_{m_k})$, ($m_k$ a selection from $\mathbb{N}$), $h_k \to h$. We must show that $h \in \mathbb{H}(V)$. We shall prove in lemma 6.2 that there exists a sequence of payoff configurations $\hat{h}_k$ such that, at least for a subsequence, $(h_S)_k$ converges to an element of V(S), say h_S.
Moreover λ_k converges to an element of $\mathbb{R}_+^N - \{0\}$, say λ. We claim that $h \in \mathbb{H}(V)$ with $\hat{h} = (h_S)_{S \subset N}$ payoff configuration and λ associated weight vector. (H.2) is trivial. To check (H.3), put $\xi_S = 0$ if there is $i \in S$ such that $\lambda^i = 0$. For other coalitions S we obtain ξ_S by taking limits and using (H.4) and (H.5). Then (H.3) follows. Now we prove (H.1). We have to show that $h_S \in bd(V(S))$. Suppose, on the contrary, that $h_S \in int(V(S))$. Then for a small $a > 0$ we have $h_S + 3a1_S \in V(S)$. Then, by (2.1), there is $w_k \in V_{m_k}(S) : w_k \to h_S + 3a1_S$. Hence $w_k > h_S + 2a1_S$ from which $w_k > (h_S)_k + a1_S$, at least for large k. Then, in view of (G.2), $(h_S)_k \notin bd(V(S))$, a contradiction.

LEMMA 6.2. Let $V_m \to V$ for the closed convergence, $h_k \in \mathbb{H}(V_{m_k})$($m_k$ a selection from $\mathbb{N}$) and $h_k \to h$. Then there is a payoff configuration $\hat{h}_k$

associated to h_k, for each k, such that $\{(h_S)_k: k \in \mathbb{N}\}$ is a bounded set for each $S \in 2^N$.

Proof. Let $(h_N)_k = h_k$. Then $\{(h_N)_k: k \in \mathbb{N}\}$ is bounded. So we can concentrate on coalitions $S \neq N$. By theorem 4.3 $\hat{h}_k$ can be chosen such that $(h_S)_k$ is individually rational for each $S \neq N$. Let $v_k = (v_k(1), \dots v_k(n))$. Then $(h_S)_k \geq (v_S)_k > v_S - 1_S$ implying that $\{(h_S)_k : k \in \mathbb{N}\}$ is a lower bounded set. Upper boundedness easily follows from (G.3), so the lemma is proved.

As far as lower semicontinuity of $\mathbb{H}$ is concerned, the following example is illustrative.

EXAMPLE 6.3. Let $N = \{1,2,3\}$, $v_m(i) = 0$ for $i \in N$,

$V_m(\{1,2\}) = \{(x,y,0): x,y \geq 0,\ x + y \leq 20\} + \mathbb{R}_-^{\{1,2\}}$, $V_m(\{1,3\}) = \{(x,0,z): x,z \geq 0, 3x + z \leq 480\} + \mathbb{R}_-^{\{1,3\}}$, $V_m(\{2,3\}) = \{(0,y,z): y,z \geq 0,\ 3y + z \leq 480\} + \mathbb{R}_-^{\{2,3\}}$, $V_m(N) = \{(x,y,z): x,y,z \geq 0, 3x + y + z \leq 510,\ (2 + \frac{1}{m})x + (2 - \frac{1}{m})y + z \leq 510\} + \mathbb{R}_-^N$.

The sequence of games V_m (belonging to $\hat{\Gamma}$) converges, in Hausdorff sense, to the game $v \in \hat{\Gamma}$, such that $V(S) = V_m(S)$ for $S \neq N$,

$$V(N) = \{(x,y,z) \in \mathbb{R}_+^N : 3x + y + z \leq 510,\ 2x + 2y + z \leq 510\} + \mathbb{R}_-^N.$$

In a similar way as in the example 5.3 we can see that the vector $(58\frac{1}{3}, 55, 280)$ (associated to $\lambda = (3,1,1)$) is the unique solution for the games V_m, for each $m \in \mathbb{N}$. The limit game, on the other hand, has also the Harsanyi solution $(56\frac{1}{3}, 56\frac{1}{3}, 284\frac{2}{3})$(associated to $\lambda = (2,2,1)$). So $\mathbb{H}$ is not lower semicontinuous.

REMARK 6.4. Harsanyi [6] suggests a kind of bargaining process to select a unique outcome for games with more than one solution. Our example shows that the unique final outcome of this procedure does not have in general the continuity property.

REMARK 6.5. The previous example concerning lower semicontinuity needs several calculations because, among other conditions, we wanted only non degenerate solutions. If we allow degenerate solutions, then it is possible to construct a much easier example : take $N = \{1,2\}$ $v_m(i) = 0$ for $i \in N$, $V_m(N) = \{(1, \frac{1}{m})\} + \mathbb{R}^N_-$,

$V(N) = \{(1,0)\} + \mathbb{R}^N_-$. All points $(x,0)$ with $x \leq 1$ are solutions of V associated to $\lambda = (0,1)$, while V_m has the unique solution $(1, \frac{1}{m})$, associated to $\lambda = (\frac{1}{m}, 1)$.

To conclude our discussion on stability, we remark that in [9] it is claimed that both values have closed graph,with Hausdorff convergence. Then our results generalize to closed convergence as far as Harsanyi value is concerned; on the other hand,as we saw, to state the closed graph property of Shapley value, with Hausdorff convergence some additional assumptions are needed(10).

7. The existence theorem

All the properties we proved for the Harsanyi value are now used in order to provide the following existence theorem:

THEOREM 7.1. Let V be a NTU game in Γ such that V(S) is a convex set for each $S \in 2^N$. Then $\mathbb{H}(V) \neq \phi$.

Proof. Define, for each $m \in \mathbb{N}$, the game V_m by :

$$V_m(S) = V(S) \cap \{x \in \mathbb{R}^S : x^i \leq m \text{ for each } i \in S\}.$$

Then $V_m(S)$ is a convex set and $V_m \in \hat{\Gamma}$ for all large m. Hence, by [20] theorem 6.6 $\mathbb{H}(V_m) \neq \phi$ for all large m. From theorem 4.3 we can find a Harsanyi solution h_m and an associated payoff configuration $\hat{h}_m$ which are equibounded. Taking a subsequence, if necessary, h_m and $\hat{h}_m$ have a limit, say h and $\hat{h}$. Then $h \in \limsup \mathbb{H}(V_m)$. By theorem 6.1 we can conclude that $h \in \mathbb{H}(V)$. This completes the proof.

(10) We made additional assumptions in order to ensure $v_{\lambda_n}(S_n) \to v_\lambda(S)$ for every $\lambda_n \to \lambda$ and $S_n \to S$. It is not difficult to see that this is true if (and only if) the function : $\lambda \to v_\lambda(S)$ is (upper semi-) continuous, (on its effective domain), for every S. But for a support function this is not true in general.

8. Conclusion

We analyzed the properties of individual rationality and stability for both values, with particular attention to the degenerate case. As we saw, the properties we investigated are often used as axioms in order to characterize solutions in the cooperative setting. In their exchange, both Aumann and Roth agree on the fact that the values merit further study, (cf. [1] p. 675 and [24] p. 984); furthermore Hart, who in [9] makes a systematic comparison of the relevant properties of the values, observes that the two values "satisfy axioms that appear in form almost identical ...(it) is just the space to which the solutions belong (that) changes". We hope to give with this paper a further contribution to the understanding of the two values, showing in particular some relevant differences. We remark that our analysis of degenerate solutions underlines the fundamental role that the intermediate coalitions have in the construction of the Harsanyi value, while λ-transfer seems to use them only as a tool to establish the final result, despite of credibility (see [7] p.1306). This is naturally and perfectly reflected by the difference in the axiomatization [7] previously recalled: on the one hand only the final outcome, on the other hand all the payoff configuration. Our conclusion is that the Harsanyi value seems to be more persuasive in some respects,(see also [8]), though we agree with Hart ([7] p. 1296 and 1308) that it is less tractable and that the right framework for the application of the λ-transfer value is probably that of large games, where it provides natural solutions (see [2] and references therein).

9. References

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